TOPOLOGICAL MEANS

By

ARTHUR CRUMMER

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Abstract of Dissertation Presented to the
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By

Arthur Crummer

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Chairman: Professor Philip Bacon
Major Department: Mathematics

If X is a topological space and n is a positive integer, then X is said to admit an n-mean provided there is a continuous function $m : X^n \to X$ such that $m(x,\ldots,x) = x$ for each element x of X and such that $m(x_1,\ldots,x_n)$ is invariant under permutations of $x_1,\ldots,x_n$. In Section 1, a summary of known results is given. In Section 2, we define what is called a uniform n-mean, show that a space admits a uniform 2-mean if and only if it admits a 2-mean and that if a space admits a uniform n-mean ($n \geq 3$), then it admits an n-mean. It is proved that if a compact Hausdorff space is a retract of its hyperspace, then it is acyclic.

In Section 3, we characterize those solenoids which admit an n-mean (Proposition 3.5), give an example of a non-acyclic continuum which admits an n-mean for each positive integer n, and show that the admissibility of an n-mean is a strictly weaker condition than admissibility of a uniform n-mean whenever $n > 2$.

In Section 4, it is shown that the first Alexander cohomology group of any space is torsion free (integral coefficients) and this is

used to prove a technical result (Proposition 4.7) which is used for several purposes: we are able to give new examples of contractible spaces which admit no n-means, to prove that if a compact Hausdorff space admits an n-mean ($n \geq 2$), then each of its non-zero cohomology groups (integral coefficients) is uniquely n-divisible, and to prove that if a continuum X admits an idempotent continuous multiplication having a zero, then X is acyclic.

Section 5 contains a few open questions.

INTRODUCTION

What exactly is an average? What properties should it have? There are many plausible answers to this question (see, for example, Aczél [1], especially Sections 5.3 and 6.4) but the objective here is to define a certain kind, call it a 2-mean, generalize the concept a bit and study conditions under which such a thing may or may not exist.

In a paper of 1944, G. Aumann [2] defined an n-mean (n a positive integer) on a topological space X to be a continuous function $m: X^n \to X$ such that $m(x,x,\ldots,x) = x$ for each $x \in X$ and such that $m(x_1,\ldots,x_n)$ is invariant under permutations of $x_1,\ldots,x_n$. He proved that there can be no n-mean $(n \geq 2)$ on the k-sphere, where $k \geq 1$. Now the intuitive feeling one might supply to justify this fact is that the "hole" in S^k is responsible for this turn of events; however, an example (the n-solenoid) mentioned by Eckmann [9] shows that a space may admit an n-mean in spite of "having holes." Furthermore, Bacon [5] gives an example of a contractible space (no holes) which admits an n-mean for no $n \geq 2$.

What exactly does cause a space to admit or fail to admit means?

SECTION 1 MEANS ON SPACES

The theorems stated in this section are previously known results (except for Theorem 1.10) which will be assumed in the succeeding two sections. It will be noted, however, that all the theorems of this section follow from results of Section 4 and that the proofs in Section 4 are independent of the preceding sections.

Definition 1.1

A 2-mean on a topological space X is a continuous function $m: X \times X \to X$ such that

(i) $m(x,x) = x$ for each $x \in X$,

and (ii) $m(x,y) = m(y,x)$ for all $x, y \in X$.

In this case, X is said to admit the 2-mean m.

Examples

The real numbers R admit the 2-mean $m: R \times R \to R$ defined by either of the rules

(1) $m(x,y) = (x+y)/2$ (Arithmetic Mean)

or (2) $m(x,y) = \inf\{x,y\}$.

The positive reals admit the 2-mean m defined by (1) or (2) above or by

(3) $m(x,y) = \sqrt{x \cdot y}$ (Geometric Mean) .

The closed unit interval $I = \{x \in R \mid 0 \leq x \leq 1\}$ admits the 2-mean $m: I \times I \to I$ defined by (1), (2) or (3) above or by

$$(4) \quad m(x,y) = \begin{cases} 2xy/(x+y) & \text{if } x + y \neq 0 \\ 0 & \text{if } x = 0 = y \end{cases} \quad \text{(Harmonic Mean).}$$

The concept of a 2-mean on a space generalizes in a natural way to a mean of n variables where n is any positive integer (Eckmann's definition).

Definition 1.3

A space X admits an n-mean provided there exists a continuous function $m: X^n \to X$ such that

(1) $m(x_1, x_2, \ldots, x_n) = m(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for every permutation π of $\{1, \ldots, n\}$ and for every $(x_1, \ldots, x_n) \in X^n$

and (2) $m(x, x, \ldots, x) = x$ for each $x \in X$.

In this case, we say m is an n-mean on X. For $n = 1$, note that every space admits a uniquely determined 1-mean (the identity map); consequently, our discussion of n-means will be restricted to $n \geq 2$.

For a fixed n, no characterization of spaces admitting n-means is known, but there are many partial results.

Theorem 1.4 (Bacon [4])

If X is a compact subset of R^p $(p \geq 1)$ and X admits an n-mean for some $n \geq 2$ then $R^p \setminus X$ is connected.

If X is a compact locally connected subset of R^2 and $n \geq 2$, then X admits an n-mean if and only if $R^2 \setminus X$ is connected.

(See also Proposition 2.10) below.

Definition 1.5

If G is a group and n a positive integer, then G is said to be n-divisible provided the map $g \mapsto ng$ is onto; G is divisible provided it is n-divisible for all $n \geq 1$. G is uniquely n-divisible provided the map $g \mapsto ng$ is an automorphism of G. A homomorphic n-mean on a topological group G is an n-mean $m: G^n \to G$ such that $m(x_1+y_1, \ldots, x_n+y_n) = m(x_1, \ldots, x_n) + m(y_1, \ldots, y_n)$ whenever $x_1, \ldots, x_n, y_1, \ldots, y_n \in G$.

Notice that a divisible group is torsion free if and only if it is uniquely n-divisible for each $n \geq 1$.

If G is a uniquely n-divisible group, a homomorphic n-mean may be defined on G by the assignment $(x_1, \ldots, x_n) \mapsto \frac{1}{n}(x_1+x_2+\ldots+x_n)$ where $\frac{1}{n}$ denotes the inverse of the automorphism $g \mapsto ng$ of G. Eckmann has shown that these are the only homomorphic n-means.

Theorem 1.6 (Eckmann [9], p. 334)

If a group G admits a homomorphic n-mean M, then

(1) G is Abelian,

(2) G is uniquely n-divisible, and

(3) M satisfies $M(x_1, \ldots, x_n) = \frac{1}{n}(x_1+\ldots+x_n) \qquad (x_i \in G)$.

(See also Proposition 4.2 below.)

Theorem 1.7 (Eckmann [9])

If a space X admits an n-mean, then the fundamental group of X, $\pi(X)$, admits a homomorphic n-mean.

(See also Proposition 4.3 below.)

Theorem 1.8 (Sigmon [19])

Suppose X is a continuum and R a principal ideal domain such that $H^p(X;R)$ is torsion free as an R-module for all $p \geq 1$. If X admits an n-mean, then $H^p(X;R)$ is uniquely n-divisible for all $p \geq 1$.

We remark that if R is the integers, the hypothesis of Theorem 1.8 can be weakened to X compact T_2 and no assumptions on $H^p(X;Z)$ and still draw the same conclusions.

(See Proposition 4.13.)

Corollary 1.9

If a compact T_2 space X admits an n-mean, then $H^p(X;Z_n) = 0$ for all $p \geq 1$. (Z_n is the integers mod n.)

(See also Corollary 4.12 below.)

Theorem 1.10

If X is a continuum and there exists an idempotent continuous map $\mu: X \times X \to X$ having a zero, then $H^p(X;Z) = 0$ for all $p \geq 1$.

This theorem will be proven in Section 4. See Proposition 4.14.

SECTION 2 HYPERSPACES AND MEANS

If X is a T_1-space, then the hyperspace of X, denoted 2^X, is the space whose elements are the nonvoid closed subsets of X and whose topology is generated by sets of the form

$$\langle U_1, \ldots, U_n \rangle = \{C \in 2^X \mid C \subset \bigcup_{i=1}^{n} U_i \text{ and } C \cap U_i \neq \emptyset \text{ for each } i \leq n\}$$

where n is a positive integer and each of $U_1, \ldots, U_n$ is open in X. For each positive integer n, $\mathcal{F}_n(X)$ will denote the subspace $\{C \in 2^X \mid C$ has at most n elements$\}$; note that we have a tower $\mathcal{F}_1(X) \subset \mathcal{F}_2(X) \subset \ldots$ whose union $\mathcal{F}(X) = \bigcup_{n=1}^{\infty} \mathcal{F}_n(X)$ is dense in 2^X. The embedding $x \mapsto \{x\}$ will henceforth be used to regard X as a subspace of 2^X. It is well known that 2^X is compact and T_2 if and only if X is compact and T_2; also, 2^X is connected if and only if X is connected. See Michael [16], Theorems 4.2, 4.9, 4.10.

Definition 2.1

A T_1-space is said to admit a uniform n-mean provided X is a retract of $\mathcal{F}_n(X)$. In this case, a retraction $r: \mathcal{F}_n(X) \to X$ is called a uniform n-mean (on X).

In the following lemma, (i) was stated in [16] without proof, (ii) was stated in [7], Theorem 2, for each $k \geq 2$ but is false for $k \geq 3$ ([11], p. 308), and in (iii) the stronger conclusion that f is a closed map holds [11].

Lemma 2.2

If X is a T_1-space, k is a positive integer and if $f: X^k \to \mathcal{F}_k(X)$ is defined by $f(x_1,\dots,x_k) = \{x_1,\dots,x_k\}$, then

(i) f is continuous,

(ii) if $k = 2$, then f is an open (hence quotient) map, and

(iii) if X is T_2, then f is a quotient map.

Proof:

(i) Assuming that each of $U_1,\dots,U_n$ is open in X, note that

$$f^{-1}(\langle U_1,\dots,U_n\rangle \cap \mathcal{F}_k(X)) = \{(x_1,\dots,x_k) \in X^k \mid \{x_1,\dots,x_k\} \in \langle U_1,\dots,U_n\rangle\}$$

$= \{(x_1,\dots,x_k) \in X^k \mid$ for all $j \in \{1,\dots,k\}$, $x_j \in \bigcup_{i=1}^{n} U_i$,

and for all $i \in \{1,\dots,n\}$, there exists $j \in \{1,2,\dots,k\}$

such that $x_j \in U_i\}$

$= \left(\bigcup_{i=1}^{n} U_i\right)^k \cap \bigcap_{i=1}^{n} \bigcup_{j=1}^{k} \pi_j^{-1}(U_i)$, so f is continuous.

(ii) Now to see that $f: X^2 \to \mathcal{F}_2(X)$ is open, suppose that $U_1 \times U_2$ is a basic open set in X^2. We shall show that $f(U_1 \times U_2) = \langle U_1,U_2\rangle \cap \mathcal{F}_2(X)$; indeed, if $(u_1,u_2) \in U_1 \times U_2$, then $f(u_1,u_2) = \{u_1,u_2\}$ has at most two elements and meets each of U_1 and U_2 and is contained in $U_1 \cup U_2$ so that $f(u_1,u_2) \in \langle U_1,U_2\rangle \cap \mathcal{F}_2(X)$. On the other hand, if $A \in \langle U_1,U_2\rangle \cap \mathcal{F}_2(X)$, then A contains either one or two points. But if A has exactly one point a, then $a \in U_1 \cap U_2$ and hence $A = \{a\} = f(a,a) \in f(U_1 \times U_2)$. If A has exactly two points, then we consider two cases. Case I if $(U_2 \setminus U_1) \cap A = \emptyset$, then let $a_2 \in U_2 \cap A$ and let a_1 be the other point of A; then $A = f(a_1,a_2) \in f(U_1 \times U_2)$.

Case II if $(U_2 \setminus U_1) \cap A \neq \emptyset$, then we let $a_2 \in (U_2 \setminus U_1) \cap A$, choose $a_1 \in U_1 \cap A$, and note that $A = f(a_1, a_2) \in f(U_1 \times U_2)$. It follows that f is an open map.

(iii) Supposing that $\mathcal{A} \subset \mathcal{F}_k(X)$ is such that $f^{-1}(\mathcal{A})$ is open in X^k, we wish to argue that $\mathcal{A}$ is open. But if $A \in \mathcal{A}$, we may choose $(x_1, \ldots, x_k) \in X^k$ such that $f(x_1, \ldots, x_k) = A$ and then (using the T_2 assumption and the openness of $f^{-1}(\mathcal{A})$) choose open sets $U_1, \ldots, U_k$ in X such that $(x_1, \ldots, x_k) \in U_1 \times \ldots \times U_k \subset f^{-1}(\mathcal{A})$ and $\{U_1, \ldots, U_k\}$ is pairwise disjoint. But then $A = f(x_1, \ldots, x_k) \in f(U_1 \times \ldots \times U_k) = \langle U_1, \ldots, U_k \rangle \cap \mathcal{F}_k(X) \subset \mathcal{A}$.

Proposition 2.3

If a T_1-space admits a uniform n-mean, then it admits a uniform k-mean for each $k \leq n$. If a T_1-space admits a uniform k-mean, then it admits a k-mean. A T_1-space admits a uniform 2-mean if and only if it admits a 2-mean.

Proof:

The restriction of a uniform n-mean $r: \mathcal{F}_n(X) \to X$ to $\mathcal{F}_k(X)$ is a uniform k-mean. Supposing $r: \mathcal{F}_k(X) \to X$ is a uniform k-mean on X, define $m: X^k \to X$ by $m(x_1, \ldots, x_k) = r(\{x_1, \ldots, x_k\})$ and note that m is symmetric and idempotent. Since m is the composition $X^k \xrightarrow{f} \mathcal{F}_k(X) \xrightarrow{r} X$ (f as in Lemma 2.2), it is continuous.

Now suppose a T_1-space admits a 2-mean $m: X^2 \to X$. Then the map $r: \mathcal{F}_2(X) \to X$ such that $r(\{x_1, x_2\}) = m(x_1, x_2)$ is well defined and satisfies $r(x) = r(\{x, x\}) = m(x, x) = x$ for all $x \in X$. Since

$f: X^2 \to \mathcal{F}_2(X)$ is a quotient map (Lemma 2.2 (ii)) and $r \circ f = m$ is continuous, r is continuous. We conclude that r is a uniform 2-mean on X.

For $n > 2$, the n-solenoid Σ_n (discussed in Section 3 below) provides an example of a continuum which admits an n-mean but does not admit a uniform n-mean (Corollary 3.8).

Notice that if $n \geq 2$ and if an n-mean is obtained from a uniform n-mean $r: \mathcal{F}_n(X) \to X$ by composition $m = r \circ f: X^n \to \mathcal{F}_n(X) \to X$, then m is insensitive to repetition in the sense that $m(x_1, \ldots, x_n) = m(y_1, \ldots, y_n)$ whenever $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$. Furthermore, if either $n = 2$ or X is T_2, then any n-mean $m: X^n \to X$ which is insensitive to repetition in the above sense can be factored $m = r \circ f$ where $r: \mathcal{F}_n(X) \to X$ is a uniform n-mean. Can this always be done?

<u>Proposition 2.4</u>

If X is a T_1-space, then 2^X admits a uniform n-mean for each $n \geq 2$.

<u>Proof:</u>

If $n \geq 2$, then the map $r: \mathcal{F}_n(2^X) \to 2^X$ defined by $r(\{A_1, \ldots, A_n\}) = A_1 \cup \ldots \cup A_n$ is continuous (Michael [16]; Theorem 5.7.2) and satisfies $r(\{A\}) = A$ for each $A \in 2^X$.

<u>Proposition 2.5</u>

If the compact T_2 space X has a finite number of components, then 2^X is acyclic.

Proof:

If $X_1, X_2, \ldots, X_n$ denote the components of X and if we adjoin $\{\emptyset\}$ to each of the spaces 2^X, $2^{X_1}, \ldots, 2^{X_n}$ as an isolated point, then the function $\psi: 2^X \cup \{\emptyset\} \to (2^{X_1} \cup \{\emptyset\}) \times \ldots \times (2^{X_n} \cup \{\emptyset\})$ defined by $\psi(F) = (F \cap X_1, \ldots, F \cap X_n)$ is a homeomorphism. Now if K is one of the $(2^n - 1)$ components of 2^X and if $C = \psi(K)$, then since C is a component of $(2^{X_1} \cup \{\emptyset\}) \times \ldots \times (2^{X_n} \cup \{\emptyset\})$, we must have $C = C_1 \times C_2 \times \ldots \times C_n$ where for each i, C_i is a component of $2^{X_i} \cup \{\emptyset\}$, i.e., C_i is either $\{\emptyset\}$ or 2^{X_i}. Define $\mu: C \times C \to C$ by the rule $\mu((A_1, \ldots, A_n), (B_1, \ldots, B_n)) = (A_1 \cup B_1, \ldots, A_n \cup B_n)$ and note that, for each i, the diagram below is analytic; so μ is continuous and idempotent. Furthermore, since the element $(Y_1, \ldots, Y_n) \in C$ defined by

$$Y_i = \begin{cases} \emptyset & \text{if } C_i = \{\emptyset\} \\ X_i & \text{if } C_i = 2^{X_i} \end{cases}$$

$$\begin{tikzcd} C \times C \arrow[r, "\mu"] \arrow[d, "\pi_i \times \pi_i"] & C \arrow[d, "\pi_i"] \\ C_i \times C_i \arrow[r, "\text{Union}"] & C_i \end{tikzcd}$$

is a zero for μ, we conclude (by Theorem 1.10) that the continuum C is acyclic and hence K is acyclic. Since this is true for every component K of 2^X, 2^X is acyclic.

Corollary 2.6

If the compact T_2 space X is either connected or locally connected, then 2^X is acyclic.

We make the observation here that if X is a compact T_2 space such that $H^1(2^X; Z) = 0$, then every map $f: 2^X \to S^1$ is null homotopic (Huber [13]); therefore, if a compact T_2 space X has finitely many

components, then 2^X satisfies property (b) as defined by Whyburn [22]. This is an alternate proof (assuming weaker hypotheses) of a result of Nadler [17]: the hyperspace of a continuum has property (b).

Nadler [17] raises the question "what are necessary and sufficient conditions that a continuum X be a retract of 2^X?" and in the case that X is a locally connected metric continuum, observes that (by [14], 2^X is an absolute retract) X is a retract of 2^X if and only if X is an absolute retract. Proposition 2.8 below is a necessary condition for a compact T_2 space to be a retract of its hyperspace.

Lemma 2.7

If X is a retract of its hyperspace, then every component of X is a retract of its hyperspace.

Proof:

Suppose C is a component of X and $r:2^X \to X$ is a retraction. Now the relative topology on the subset 2^C of 2^X is the same as the hyperspace topology (see definitions and note that $\langle U_1,\dots,U_n\rangle \cap 2^C = \langle U_1 \cap C,\dots,U_n \cap C\rangle$) and, since C is connected, so is 2^C. Consequently, $r[2^C]$ is a connected subset of X containing the component C so that $r[2^C] = C$. Letting $r':2^C \to C$ be defined as the restriction of r, we conclude that C is a retract of 2^C.

Proposition 2.8

If a compact T_2 space X is a retract of its hyperspace, then

(1) X admits a uniform n-mean (hence an n-mean) for all $n \geq 2$,

and (2) X is acyclic.

Proof:

(1) is obvious. If C is any component of X, then 2^C is acyclic (by 2.6) and since C is a retract of 2^C (lemma), C is also acyclic. But since this is true of each component, X is acyclic.

A result of Eckmann, Ganea and Hilton ([10], Satz 6′) is that any compact polyhedron which admits an n-mean for some $n \geq 2$ is necessarily contractible. With this and Proposition 2.8 in mind, one might be tempted to conjecture that a continuum which admits an n-mean for all $n \geq 2$ should be acyclic. However, the rational solenoid Σ_ω (see Section 3) shows this to be false; hence, in particular, there is a continuum which admits all n-means but is not a retract of its hyperspace. See Problem 2 in the last section.

The next lemma is well known and is stated here for reference; see Borsuk [6], Theorem 13.1; Dugundji [8], 2.1, page 357.

Lemma 2.9

If X is a compact subset of R^n $(n \geq 1)$ and $G \neq 0$, then $R^n \setminus X$ is connected if and only if $H^{n-1}(X;G) = 0$. If X is a locally connected continuum in R^2, then $R^2 \setminus X$ is connected if and only if X is an absolute retract.

Proposition 2.10

If X is a locally connected continuum contained in R^2, then the following are equivalent:

(1) X admits an n-mean for some $n \geq 2$

(2) X admits an n-mean for each $n \geq 2$

(3) X is a retract of 2^X

(4) X is an absolute retract

(5) $R^2 \setminus X$ is connected

(6) $H^1(X;Z_n) = 0$ for some $n \geq 2$.

Proof:

It is clear that $(6) \Rightarrow (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$. By 1.9, $(1) \Rightarrow (6)$.

We now illuminate the role of connectedness.

Proposition 2.11

(1) If a space X admits an n-mean, then so does each component of X.

(2) If the components of a space X are open, then X admits an n-mean if and only if each component admits an n-mean.

(3) If X is a retract of 2^X, then each component of X is a retract of its hyperspace.

(4) If the components of a compact T_2 space X are open, then X is a retract of 2^X if and only if each component of X is a retract of its hyperspace.

Proof:

(1) This has been proved in [2]. If $m:X^n \to X$ is an n-mean and C is a component of X, then since $m(C^n)$ is connected and contains C, $m(C^n) = C$. The restriction of m to C^n is an n-mean on C.

(2) If for each component C of X, $m_C:C^n \to C$ is an n-mean, then (imitating Bacon [4]), if x_o is a fixed element of X, then an n-mean $m:X^n \to X$ is defined by

$$m(p) = \begin{cases} m_C(p) & \text{if } p \in C^n \text{ for some component } C \text{ of } X \\ x_o & \text{if } p \notin \bigcup \{C^n \mid C \text{ is a component of } X\} . \end{cases}$$

(3) Has been proven (Lemma 2.7).

(4) If $C_1, \ldots, C_k$ are the components of X and for each i, $r_i : 2^{C_i} \to C_i$ is a retraction, then for a fixed element x_o of X, we define $r: 2^X \to X$ as follows: if $A \in 2^X$, then

$$r(A) = \begin{cases} r_i(A) & \text{if } A \in 2^{C_i} \text{ for some } i \leq k \\ x_o & \text{if } A \notin \bigcup_{i=1}^{k} 2^{C_i} . \end{cases}$$

Thus r is well defined and $r(x) = x$ for each $x \in X$. To see that r is continuous, one verifies that if U is open in X, then

$$r^{-1}(U) = \begin{cases} \bigcup_{i=1}^{k} r_i^{-1}(U) & \text{if } x_o \notin U \\ \left[\bigcup_{i=1}^{k} r_i^{-1}(U)\right] \cup \left[2^X \setminus \bigcup_{i=1}^{k} 2^{C_i}\right] & \text{if } x_o \in U. \end{cases}$$

But each $r_i^{-1}(U)$ is open in 2^{C_i} and $2^{C_i} = \langle C_i \rangle$ is open in 2^X (because C_i is open in X), so that $r_i^{-1}(U)$ is open in 2^X. Also, since each C_i is compact, each 2^{C_i} is compact and hence closed in 2^X; therefore, $2^X \setminus \bigcup_{i=1}^{k} 2^{C_i}$ is open in 2^X. Thus, in either case, $r^{-1}(U)$ is open in 2^X.

<u>Corollary 2.12</u>

If X is a locally connected compact T_2 space, then

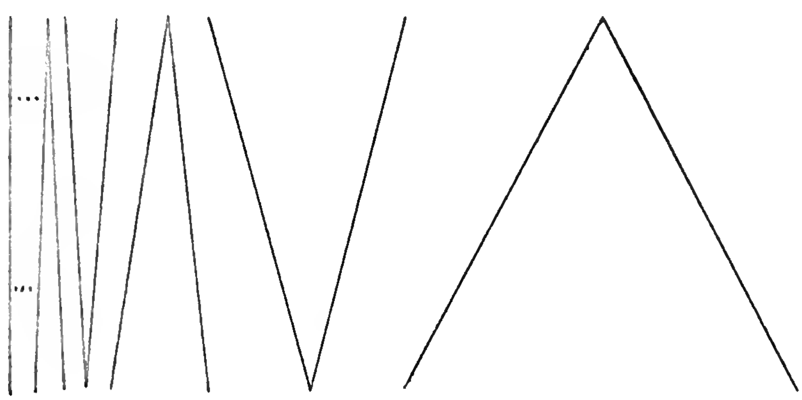
(1) X admits an n-mean if and only if each component admits an n-mean,

(2) X is a retract of 2^X if and only if each component is a retract of its hyperspace.

When can the conclusions of Corollary 2.12 be drawn assuming that X is a <u>non</u> locally connected compact T_2 space? Sometimes they can, for the space $X = \{(x,y) \in R^2 \mid x = 0 \text{ or } x = \frac{1}{n} \text{ for some positive integer } n \text{ and } 0 \leq y \leq 1\}$ is a retract of its hyperspace (A retraction $r:2^X \to X$ is defined by $r(A) = (\inf (\pi_1(A)), \inf \pi_2(A).)$ while the same is true of each of its components. On the other hand, sometimes they cannot, because the space X of the next proposition admits no 2-mean (hence cannot be a retract of its hyperspace), while each of its components is an absolute retract.

<u>Proposition 2.13</u>

The space X indicated below admits no 2-mean.



This can be proven by imitating Bacon's proof [3] that the $\sin \frac{1}{x}$ continuum admits no 2-mean.

SECTION 3 SOLENOIDS AND MEANS

We shall show in this section that Sigmon's necessary condition for a compact space X to admit an n-mean ($n \geq 2$), $H^1(X;Z_n) = 0$, is also sufficient provided X is a solenoid and that a solenoid admits an n-mean if and only if it admits a homomorphic n-mean.

<u>Definitions 3.1</u>

If $a = (a_1, a_2, \ldots)$ is a sequence of positive integers, then the a-solenoid, denoted Σ_a, is the inverse limit of the system

$$S^1 \xleftarrow{\varphi_{a_1}} S^1 \xleftarrow{\varphi_{a_2}} S^1 \xleftarrow{\varphi_{a_3}} S^1 \xleftarrow{\varphi_{a_4}} \cdots$$

where, for each $i \geq 1$, φ_{a_i} is the map $z \mapsto z^{a_i}$. If a is such a sequence, then Σ_a is the space $\{(z_1, z_2, \ldots) \mid \text{for all } i \geq 1,\ z_i \in S^1$ and $z_{i+1}^{a_i} = z_i\}$ with the relative topology. By a solenoid, we will understand a space Σ_a where a is a sequence of positive integers. The solenoid Σ_ω corresponding to $\omega = (2,3,4,\ldots)$ is called the rational solenoid and the solenoid Σ_n corresponding to the sequence $(n,n,n,\ldots)$ is called the n-adic solenoid.

Note that for any sequence a, the solenoid Σ_a is a continuum (any inverse limit of continua is a continuum) and is an Abelian topological group under coordinatewise multiplication; hence is a divisible group (because if G is a compact Abelian topological group, then G is divisible if and only if G is connected. See Hewitt and Ross [12], Theorem 24.25).

<u>Proposition 3.2</u>

If $a = (a_1, a_2, \ldots)$ is a sequence of positive integers, then $H^1(\Sigma_a; Z) = \{k/(a_1 a_2 \ldots a_n) \mid k \in Z \text{ and } n \geq 1\}$ (subgroup of Q = rationals).

<u>Proof:</u>

$H^1(\Sigma_a; Z)$ is, by continuity of Alexander Spanier cohomology, isomorphic with the direct limit of the system

$Z \xrightarrow{k_1} Z \xrightarrow{k_2} Z \xrightarrow{k_3} Z \to \ldots$ where $k_i(n) = a_i n$ for all $i \geq 1$ and for all $n \in Z$.

<u>Corollary 3.3</u>

$H^1(\Sigma_\omega; Z) = Q$ and $H^1(\Sigma_n; Z) = \{k/n^t \mid k \in Z \text{ and } t \geq 0\}$.

If $a = (2,3,5,7\ldots)$ is the sequence of primes, then $H^1(\Sigma_a; Z) = \{q \in Q \mid q = m/n$ for some $m, n \in Z$ such that $n > 0$ and n is square free$\}$.

<u>Proposition 3.4</u>

If $a = (a_1, a_2, \ldots)$ is a sequence of positive integers and n is a positive integer, then $H^1(\Sigma_a; Z_n) = 0$ if and only if for all $k \geq 1$ there exists $\hat{k} \geq k$ such that $a_k a_{k+1} \ldots a_{\hat{k}}$ is a multiple of n. Consequently, if p divides n, then $H^1(\Sigma_a; Z_n) = 0 \Rightarrow H^1(\Sigma_a; Z_p) = 0$.

<u>Proof:</u>

$H^1(\Sigma_a; Z_n) = 0$ if and only if the identity in each copy of Z_n in the direct system below gets taken to zero by the composition of a finite number of the k_i's, where k_i is multiplication by a_i:

$Z_n \xrightarrow{k_1} Z_n \xrightarrow{k_2} Z_n \xrightarrow{k_3} Z_n \to \ldots$. The rest is clear.

Proposition 3.5

If X is a solenoid and n a positive integer, then the following are equivalent:

(1) X admits an n-mean,

(2) $H^1(X;Z_n) = 0$,

(3) $H^1(X;Z_p) = 0$ for each prime divisor p of n,

(4) X is uniquely n-divisible (i.e., X admits the homomorphic n-mean).

Proof:

Previous results show that (4) → (1) → (2) → (3). Write $n = p_1 p_2 \dots p_r$ where each p_i is prime and $X = \Sigma_a$ where $a = (a_1, a_2, \dots)$. We assume (3) and note that to prove (4) it is sufficient to show that $x^n = 1$ for no $x \in \Sigma_a \setminus \{1\}$. So suppose $x = (x_1, x_1, \dots) \in \Sigma_a$ such that $1 = x^n = (x_1^n, x_2^n, \dots)$. If $k \geq 1$, then since $H^1(\Sigma_a; Z_{p_1}) = 0$ and p_1 is prime, there exists $k_1 \geq k$ such that a_{k_1} is a multiple of p_1. But then since $H^1(\Sigma_a; Z_{p_2}) = 0$ and p_2 is prime, there exists $k_2 \geq k_1$ such that a_{k_2} is a multiple of p_2. Continuing in this manner we obtain $k \leq k_1 \leq k_2 \leq \dots \leq k_r$ such that, for each i, a_{k_i} is a multiple of p_i.

But then $x_k = x_{k+1}^{a_k} = x_{k+2}^{a_k a_{k+1}} = x_{k+3}^{a_k a_{k+1} a_{k+2}} = \dots = x_{k_r+1}^{a_k \dots a_{k_r}} = 1$

(because $a_k \dots a_{k_r}$ is a multiple of n). We have shown that $x_k = 1$ for every $k \geq 1$, hence $x = 1$.

Corollary 3.6

A solenoid admits an mn-mean if and only if it admits both an n-mean and an m-mean.

Corollary 3.7

If n is a positive integer and $a = (a_1, a_2, \ldots)$, then Σ_a admits an n-mean if and only if each prime divisor of n divides each term of some subsequence of a.

Proof:

Use (1) → (3) in Proposition 3.5 and Proposition 3.4.

Corollary 3.8

If $n > 2$, then Σ_n does not admit an $(n-1)$-mean; hence Σ_n admits an n-mean but not a uniform n-mean.

Proof:

Let p be a prime divisor of $n-1$. Then by Corollary 3.7, p divides n; hence p is a prime factor of $n - (n-1) = 1$, a contradiction.

Proposition 3.9

If $a = (a_1, a_2, \ldots)$ is a sequence of positive integers, then Σ_a admits an n-mean if and only if $H^1(\Sigma_a; Z)$ is uniquely n-divisible.

Proof:

Suppose $H^1(\Sigma_a; Z)$ is n-divisible and let p be a prime divisor of n. For any $k \geq 1$, $1/(pa_1 \ldots a_k) = m/(a_1 \ldots a_{\hat{k}})$ for some $\hat{k} \geq 1$ and $m \in Z$ (because the cohomology group is, in fact, p divisible). But then $m = a_1 \ldots a_{\hat{k}}/pa_1 \ldots a_k$ is an integer so that $\hat{k} > k$ and $m = a_{k+1} \ldots a_{\hat{k}}/p$; whence p divides a_i for some $i > k$. By Corollary 3.7, Σ_a admits an n-mean.

For necessity, we refer to Proposition 4.13 below: if the compact T_2 space X admits an n-mean, then $H^p(X;Z)$ is uniquely n-divisible.

If $\omega = (1,2,3,4\ldots)$, then Σ_ω admits all n-means $(H^1(\Sigma_\omega;Z) = Q)$ and if $a = (2,3,5,7,\ldots)$ is the sequence of primes, then Σ_a admits an n-mean for no $n \geq 2$.

SECTION 4 LOCAL PROPERTIES AND GENERALIZATIONS

Definition 4.1

A subgroup G of H is n-divisible in H provided each $g \in G$ can be expressed $g = nh$ for some $h \in H$ (i.e., $G \subset nH$). G is uniquely n-divisible in H provided for each $g \in G$ there is exactly one $h \in H$ such that $g = nh$. If G and H are any groups, a homomorphism $f: G^n \to H$ is said to be symmetric provided $f(g_1, g_2, \ldots, g_n)$ is invariant under permutations of $g_1, \ldots, g_n$.

Proposition 4.2

If G and H are groups and $M: G^n \to H$ is a symmetric homomorphism, then the map $\mu: G \to H$ defined by $\mu(x) = M(x, 0, 0, \ldots, 0)$ is a homomorphism such that

(1) $\mu(G)$ is Abelian,

(2) the diagram below is analytic where Δ is the diagonal map and "n" denotes the map $g \mapsto ng$,

$$\begin{tikzcd} G^n \arrow[r, "M"] & H \\ G \arrow[u, "\Delta"] \arrow[r, "n"] & G \arrow[u, "\mu"'] \end{tikzcd}$$

and (3) $M\Delta(G)$ is n-divisible in $\mu(G)$ (hence in H).

Proof:

μ is a homomorphism because $\mu(x+y) = M(x+y,0,\ldots,0) = M(x,0,0,\ldots,0) + M(y,0,\ldots,0) = \mu(x) + \mu(y)$. $\mu(G)$ is Abelian, for if $x,y \in G$, then $\mu(x) + \mu(y) = M(x,0,\ldots,0) + M(y,0,\ldots,0) = M(x,0,0,\ldots,0) + M(0,y,0,0,0) = M(x,y,0,\ldots,0) = M(y,x,0,\ldots,0) = \ldots = \mu(y) + \mu(x)$.

Now if $x \in G$, then $M\Delta(x) = M(x,x,x,\ldots,x) = M(nx,0,\ldots,0) = \mu(nx) = n\mu(x)$, so the diagram is analytic and $M\Delta(x) = n\mu(x) \in n\mu(G) \subset nH$.

In case $G = H$ and M is a homomorphic n-mean, we have Theorem 1.6.

Proposition 4.3

If $m: X^n \to X$ is an n-mean on a space X $(n \geq 2)$ and if $\emptyset \neq Q \subset P \subset X$ such that $m[Q^n] \subset P$, then there exist homomorphisms $M: \Pi(Q)^n \to \Pi(P)$ and $\mu: \Pi(Q) \to \Pi(P)$ such that

(1) M is symmetric,

(2) $\mu[\Pi(Q)]$ is Abelian,

(3) the diagram below is analytic, where Δ is the diagonal map and i_* is induced by inclusion $i: Q \subset P$,

$$\begin{tikzcd} \Pi(Q)^n \arrow[r, "M"] & \Pi(P) \\ \Pi(Q) \arrow[u, "\Delta"] \arrow[ur, "i_*"] \arrow[r, "n"'] & \Pi(Q) \arrow[u, "\mu"'] \end{tikzcd}$$

and (4) $i_* \Pi(Q)$ is an n-divisible in $\mu\Pi(Q)$ (hence in $\Pi(P)$).

Proof:

It is tacitly assumed that all homotopy groups are relative to a fixed point in Q. Define M as follows: if $[w_1],\ldots,[w_n]$ are homotopy classes of the loops $w_1,\ldots,w_n$ in Q, then $M([w_1],\ldots,[w_n])$ is the homotopy class of the loop $t \mapsto m(w_1(t),\ldots,w_n(t))$ in P. Then M is well defined, symmetric, and satisfies $M \circ \Delta = i_*$. Now, define μ by the formula $\mu(x) = M(x,0,0,\ldots,0)$ and apply Proposition 4.2.

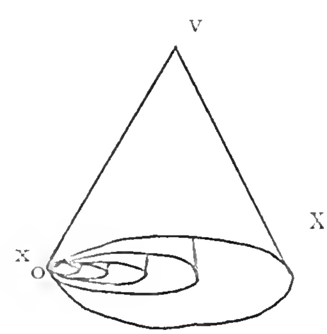
Corollary 4.4

If $X = \bigcup_{t=1}^{\infty} C_t$ where $C_t = \{p \in R^2 \mid \| p - (\frac{1}{t},0)\| = \frac{1}{t}\}$, then the cone C(X) over X admits an n-mean for no $n \geq 2$.

Proof:

Identify X with $\{(\overline{x,0}) \mid x \in X\} \subset C(X)$ in the natural way (where $(\overline{x,t})$ denotes the equivalence class of (x,t) in C(X)), let $x_o = (\overline{0,0})$, and let v denote the vertex of C(X).

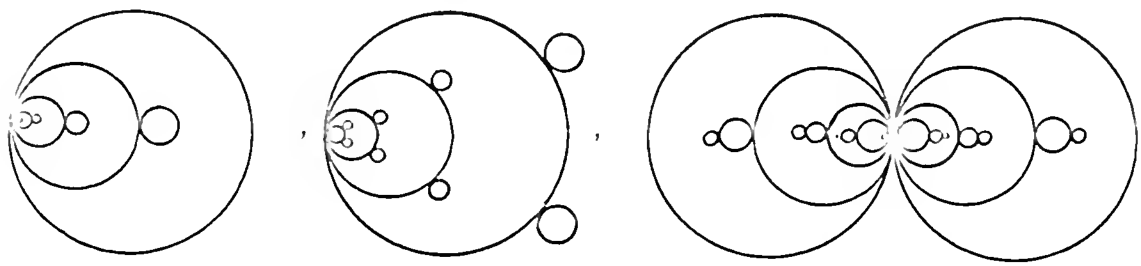
Suppose $n \geq 2$ and $m: C(X)^n \to C(X)$ is an n-mean. If $P = C(X)\setminus\{v\}$, then by continuity of m, there exists an open set U containing x_o such that $m[U^n] \subset P$. Letting $Q_k = \bigcup_{t=k}^{\infty} C_t$, we note that $\bigcap_{k=1}^{\infty} Q_k = \{x_o\} \subset U$



so that there is a $k \geq 1$ such that $Q_k \subset U$. Now $m[Q_k^n] \subset P$, so by Proposition 4.3, if $i: Q \subset P$, $i_* \pi(Q_k)$ is an Abelian group (for it is a subgroup of $\mu[\pi(Q_k)]$ where μ is the homomorphism of Proposition 4.3) and, hence, since i_* is a monomorphism, $\pi(Q_k)$ is Abelian--a contradiction.

Notice that Eckmann's results (Theorems 1.6 and 1.7 of Section 1) are special cases of Proposition 4.2 (let $G = H$) and of Proposition 4.3 (let $P = Q = X$); indeed the proofs given here are inspired by his arguments.

It is clear that minor modifications of the argument for Corollary 4.4 can be used to prove that no n-mean can be defined on the cone $C(X)$ where X is a space such as indicated by any of these:



So we have numerous contractible spaces which admit no n-means. However, the technique used here does not apply to the higher dimensional cases such as $C(X)$ where $X = \bigcup_{t \geq 1} C_t$ and C_t is the sphere

$$C_t = \{p \in R^n \mid \| p - (\tfrac{1}{t},0,0,0,\ldots,0) \| = \tfrac{1}{t}\}.$$

In order to deal with this situation, we will now set out to establish a cohomological version of Proposition 4.2. For convenience, we first state some well-known facts which will be used in the sequel. It will be assumed that the coefficient ring R is a principal ideal domain (with unit 1). See Spanier [20].

Lemma

If X and Y are compact T_2 spaces, then there is a natural exact sequence ("Künneth formula")

$$0 \to \sum_{i+j=p} H^i(X) \otimes H^j(Y) \xrightarrow{\mu} H^p(X \times Y) \to \sum_{i+j=p+1} H^i(X) * H^j(Y) \to 0.$$

If Q is any space, there exists a homomorphism ("cup product")

$$\cup : \sum_{i+j=p} H^i(Q) \otimes H^j(Q) \to H^p(Q) \quad \text{which satisfies the following:}$$

If 1 denotes the cohomology class (in $H^o(Q)$) of the constant cochain $\varphi : Q \to R$ such that $\varphi(x) = 1_R$ for all $x \in Q$, then $1 \cup h = h = h \cup 1$ for each $h \in H^p(Q)$. A homomorphism, called the "cross product,"

$$X: \sum_{i+j=p} H^i(Q) \otimes H^j(Q) \to H^p(Q \times Q) \quad \text{is defined by the formula}$$

$k^i \times k^j = \pi_1^* k^i \cup \pi_2^* k^j$ where $\pi_1, \pi_2 : Q \times Q \to Q$ are the projections. Finally, if $\Delta : Q \to Q \times Q$ is the diagonal map, then $\Delta^*(h^i \times h^j) = h^i \cup h^j$ and if Q is compact T_2, then the cross product $\times$ is the map μ in the Künneth formula.

<u>Proposition 4.5</u>

If X is any space and G is a torsion-free module over a ring R, then $H^1(X;G)$ is torsion free. In particular, $H^1(X;Z)$ is torsion free for any space X.

<u>Proof:</u>

Our assumption on G is that if $r \in R\setminus\{0\}$ and $g \in G$ such that $rg = 0$, then $g = 0$. We establish notation with these definitions: A simplicial complex K is a collection of nonempty subsets of some fixed set $V(K)$ (vertices of K) such that (1) if $v \in V(K)$, then $\{v\} \in K$, and (2) if s is a nonempty subset of some element of K, then $s \in K$. If K is a simplicial complex and $p \geq 0$, then let

$K_p = \{(v_o,\ldots,v_p) | \{v_o,\ldots,v_p\} \in K\}$ and

$C^p(K;G) = \{\varphi | \ \varphi : K_p \to G$ is a function$\}$; defining $\delta : C^p(K;G) \to C^{p+1}(K;G)$

by $\delta(\varphi)\ (v_o,\ldots,v_{p+1}) = \sum_{i=0}^{p+1} (-1)^i \varphi(v_o,\ldots,\hat{v}_i,\ldots,v_{p+1})$, we have $\delta\delta = 0$, and then define $H^p(K;G) = \operatorname{Ker}\delta/\operatorname{Im}\delta$. Furthermore, if $\varphi \in \operatorname{Ker}\delta$, $[\varphi]$ will denote the class of φ in $H^p(K;G)$. We will use the obvious correspondence to identify K_o with $V(K)$.

Define an equivalence relation $\sim$ on $V(K)$ as follows: If $v,v' \in V(K)$, then $v \sim v'$ if and only if there exists a finite sequence $v_o,v_1,\ldots,v_n \in V(K)$ such that $v_o = v$, $v_n = v'$ and $\{v_i,v_{i+1}\} \in K$ for each $i \in \{0,1,\ldots,n-1\}$. Equivalence classes will be called "components of K."

We now begin the proof by noting that $H^p(X;G)$ is the direct limit of a system of groups $H^p(K;G)$ where each K is a simplicial complex (see Spanier [20], p. 359), and since the direct limit of torsion-free modules is torsion free, we need only prove that if K is a simplicial complex and G is a torsion-free R-module, then $H^1(K;G)$ is torsion free.

So suppose $[\varphi] \in H^1(K;G)$ and $r \in R\setminus\{0\}$ such that $r[\varphi] = 0$. Then there is a function $\psi : V(K) \to G$ such that $r\varphi = \delta\psi$; i.e., if $\{v,v'\} \in K$, then $r\varphi(v,v') = \psi(v') - \psi(v)$. We wish to define a function $\xi : V(K) \to G$ such that if $\{v,v'\} \in K$, then $\varphi(v,v') = \xi(v') - \xi(v)$. This will be done by defining ξ on each component of K. Let C be a component of K and choose $v_o \in C$. Set $\xi(v_o) = \psi(v_o)$ and define $\xi : C \to G$ as follows: if $v \in C$, then choose a finite sequence $v_o,v_1,\ldots,v_n$ such that $v_n = v$ and $\{v_i,v_{i+1}\} \in K$ for $i < n$, and define $\xi(v) = \psi(v_o) + \varphi(v_o,v_1) + \ldots + \varphi(v_{n-1},v_n)$.

We claim that (A) ξ is well defined, and (B) if $\{v,v'\} \in K$, then $\varphi(v,v') = \xi(v') - \xi(v)$.

To prove (A) suppose $v_o, w_1, w_2, \ldots, w_k = v$ is another finite sequence. To prove $\psi(v_o) + \varphi(v_o,w_1) + \varphi(w_1,w_2) + \ldots + \varphi(w_{k-1},w_k) = \psi(v_o) + \varphi(v_o,v_1) + \ldots + \varphi(v_{n-1},v_n)$, it is sufficient to show that $r[\varphi(v_o,w_1) + \varphi(w_1,w_2) + \ldots + \varphi(w_{k-1},w_k)] = r[\varphi(v_o,v_1) + \varphi(v_1,v_2) + \ldots + \varphi(v_{n-1},v_n)]$, but the left side is $(\psi(w_1) - \psi(v_o)) + (\psi(w_2) - \psi(w_1)) + \ldots + (\psi(w_k) - \psi(w_{k-1})) = \psi(w_k) - \psi(v_o) = \psi(v) - \psi(v_o)$, while the right side is $(\psi(v_1) - \psi(v_o)) + (\psi(v_2) - \psi(v_1)) + \ldots + (\psi(v_n) - \psi(v_{n-1})) = \psi(v_n) - \psi(v_o) = \psi(v) - \psi(v_o)$. To prove (B), suppose $\{v,v'\} \in K$ and choose two sequences $v_o, v_1, \ldots, v_n = v$ and $v_o, v_1', v_2', \ldots, v_k' = v'$ (with $\{v_o,v_1\} \in K$, $\{v_1,v_2\} \in K$, $\{v_o,v_1'\} \in K$, $\{v_1',v_2'\} \in K$, etc.) and note that $r(\xi(v) - \xi(v')) = r(\varphi(v_k',v_{k-1}') + \varphi(v_{k-1}',v_{k-2}') + \ldots + \varphi(v_1',v_o) + \varphi(v_o,v_1) + \varphi(v_1,v_2) + \ldots + \varphi(v_{n-1},v_n)) = (\psi(v_{k-1}') - \psi(v_k')) + (\psi(v_{k-2}') - \psi(v_{k-1}')) + \ldots + (\psi(v_{n-1}) - \psi(v_{n-2})) + (\psi(v_n) - \psi(v_{n-1})) = \psi(v_n) - \psi(v_k') = \psi(v) - \psi(v') = r(\varphi(v',v))$. We conclude that $\varphi(v',v) = \xi(v) - \xi(v')$.

Having defined ξ on each component, we have a function $\xi : V(K) \to G$ such that $\{v,v'\} \in K \Rightarrow \xi(v) - \xi(v') = \varphi(v',v)$, i.e., $\varphi = \delta\xi$; therefore, $[\varphi] = 0$.

<u>Proposition 4.6</u>

Suppose Q is a continuum, P any space, $m : Q \times Q \to P$ continuous and suppose further that $p \geq 1$ such that

either (1) $H^i(Q)$ is torsion free whenever $0 \leq i < p$

or (2) $H^i(Q) = 0$ whenever $0 < i < p$.

Then there exist homomorphisms $m_1^*, m_2^* : H^p(P) \to H^p(Q)$ such that if $h \in H^p(P)$, then $m^*(h) \in H^p(Q \times Q)$ has a unique representation in the form $m^*(h) = m_1^*(h) \times 1 + 1 \times m_2^*(h) + \mu(\theta)$ where $\theta \in \sum_{\substack{i+j=p \\ i,j>0}} H^i(Q) \otimes H^j(Q)$. [$\mu$ is the monomorphism in the Künneth formula]

Furthermore, m_1^* is induced by $m_1 : Q \to P$ such that $m_1(x) = m(x,q)$ ($q \in Q$ arbitrary) and m_2^* is induced by $m_2 : Q \to P$ such that $m_2(x) = m(q,x)$ ($q \in Q$ arbitrary).

<u>Proof</u>:

Since Q is connected, $H^o(Q) = R$ so $H^o(Q) \otimes H^p(Q) = H^p(Q)$ where the isomorphism is $1 \otimes k \mapsto k$. Thus every element of $H^o(Q) \otimes H^p(Q)$ has a unique representation in the form $1 \otimes k$ where $k \in H^p(Q)$.

Let $q \in Q$ be fixed and define $\alpha_1, \alpha_2 : Q \to Q \times Q$ by $\alpha_1(x) = (x,q)$ and $\alpha_2(x) = (q,x)$. We claim that if $k \in H^p(Q)$, then $\alpha_2^*(1 \times k) = k = \alpha_1^*(k \times 1)$. To see this, suppose k is the cohomology class of $\varphi : Q^{p+1} \to R$. Then φ is the composition $Q^{p+1} \xrightarrow{\alpha_1^{p+1}} (Q \times Q)^{p+1} \xrightarrow{\pi_1^{p+1}} Q^{p+1} \xrightarrow{\varphi} R$ so that $k = [\varphi] = [\varphi \circ \pi_1^{p+1} \circ \alpha_1^{p+1}] = \alpha_1^*[\varphi \circ \pi_1^{p+1}]$
$= \alpha_1^*(\pi_1^*[\varphi]) = \alpha_1^*(\pi_1^*[\varphi) \cup 1) = \alpha_1^*(\pi_1^*[\varphi] \cup \pi_2^*(1)) = \alpha_1^*([\varphi] \times 1)$
$= \alpha_1^*(k \times 1)$. Similarly, $k = \alpha_2^*(1 \times k)$. Now the hypothesis (1) or (2) together with Proposition 4.5 assures that $\sum_{i+j=p+1} H^i(Q) * H^j(Q) = 0$, so the maps μ and μ' are isomorphisms in the analytic diagram below.

$$\begin{tikzcd}
H^p(P) \arrow[r, "m^*"] & H^p(Q \times Q) \arrow[d] \arrow[ddl, "\alpha_2^*"', "\equiv"] & H^o(Q) \otimes H^p(Q) \oplus H^p(Q) \otimes H^o(Q) \oplus \sum\limits_{\substack{i+j=p \\ i>0}} H^i(Q) \otimes H^j(Q) \arrow[l, "\mu"'] \arrow[d] \\
& H^p(\{q\} \times Q) \arrow[dl, "\cong"] & H^o(\{q\}) \otimes H^p(Q) \oplus H^p(\{q\}) \otimes H^o(Q) \oplus \sum\limits_{\substack{i+j=p \\ i>0}} H^i(\{q\}) \otimes H^j(Q) \arrow[l, "\mu'"'] \\
H^p(Q) & &
\end{tikzcd}$$

Here the vertical maps are induced by $\{q\} \to Q$ and $Q \to Q$.

Now suppose $h \in H^p(P)$. Then there are unique elements $h_1, h_2 \in H^p(Q)$ and $\theta \in \sum\limits_{\substack{i+j=p \\ i>0}} H^i(Q) \otimes H^j(Q)$ such that $m^*(h) = \mu(1 \otimes h_2 + h_1 \otimes 1 + \theta)$, i.e., such that $m^*(h) = 1 \times h_2 + h_1 \times 1 + \mu(\theta)$. From the diagram we see that $\alpha_2^* m^* h = \alpha_2^*(1 \times h_2) = h_2$. But $m \circ \alpha_2 : Q \to P$ is the map m_2 in the statement of the theorem, so $m_2^*(h) = h_2$. A similar argument shows that $m_1^*(h) = h_1$.

We conclude that $m^*(h) = 1 \times m_2^*(h) + m_1^*(h) \times 1 + \mu(\theta)$ and this representation is unique.

Proposition 4.7 below can be proved by imitating the proof of Proposition 4.6 and employing the appropriate generalization of the lemma to products of several compact T_2 spaces.

<u>Proposition 4.7</u>

If Q is a continuum, P any space, n a positive integer and $m : Q^n \to P$ is continuous and if $p \geq 1$ is such that

either (1) $H^i(Q)$ is torsion free whenever $i < p$

or (2) $H^i(Q) = 0$ whenever $0 < i < p$,

then there exist homomorphisms $m_1^*, m_2^*, \ldots, m_n^* : H^p(P) \to H^p(Q)$ such that:

if $h \in H^p(P)$, then $m^*(h) \in H^p(Q \times Q)$ has a unique representation in the form $m^*(h) = m_1^*(h) \times 1 \times \ldots \times 1 + 1 \times m_2^*(h) \times 1 \times \ldots \times 1 + \ldots + 1 \times 1 \times \ldots \times 1 \times m_n^*(h) + \mu(\theta)$ where

$$\theta \in \sum_{\substack{i_1+i_2+\ldots+i_n=p \\ 0<i_1,i_2,\ldots,i_n<p}} H^{i_1}(Q) \otimes H^{i_2}(Q) \otimes \ldots \otimes H^{i_n}(Q).$$

(μ is the monomorphism in the n-fold Künneth formula.) Furthermore, for each i, m_i^* is induced by $m_i : Q \to P$ such that $m_i(x) = m(q,\ldots,q,x,q,\ldots q)$ where "x" is the i^{th} coordinate and $q \in Q$ is arbitrary.

<u>Proposition 4.8</u>

Suppose $m : X^n \to X$ is an n-mean, $0 \neq Q \subset P \subset X$ such that Q is a continuum, $m(Q^n) \subset P$ and $i : Q \subset P$. If $p \geq 1$ is such that $H^j(Q) = 0$ for $0 < j < p$, then $i^*H^p(P)$ is n-divisible in $H^p(Q)$.

<u>Proof:</u>

Let $q \in Q$ and define $m_1, m_2, \ldots, m_n : Q \to P$ by $m_1(x) = m(x,q,q,\ldots,q)$, $m_2(x) = m(q,x,q,\ldots,q),\ldots,m_n(x) = m(q,q,\ldots,q,x)$. If $h \in H^p(Q)$, then by Proposition 4.6 and the assumption that $H^j(Q) = 0$ for $0 < j < p$, we have

$$m^*(h) = m_1^*(h) \times 1 \times \ldots \times 1 + 1 \times m_2^*(h) \times 1 \times \ldots \times 1 + \ldots + 1 \times \ldots \times 1 \times m_n^*(h).$$

But then

$$i^*(h) = \Delta^* m^*(h) = m_1^*(h) \cup 1 \cup \ldots \cup 1 + 1 \cup m_2^*(h) \cup 1 \cup \ldots \cup 1 + \ldots + 1 \cup \ldots \cup 1 \cup m_n^*(h)$$

$$= m_1^*(h) + m_2^*(h) + \ldots + m_n^*(h) = n \cdot m_1^*(h) \in n \cdot H^p(Q).$$

Hence, $i^*H^p(P)$ is n-divisible in $H^p(Q)$.

Proposition 4.9

Suppose Y is a T_2 space and $x_o \in X \subset P \subset Y$ where P is open, X has codimension $\operatorname{cd} X \leq p$ and the inclusion $j : X \subset P$ induces an epimorphism $j^* : H^p(P) \to H^p(X)$. Suppose further that $\mathcal{Q}$ is a tower of continua in X such that

(1) $\cap \mathcal{Q} = \{x_o\}$,

and (2) if $Q \in \mathcal{Q}$, then there exists $Q_1 \in \mathcal{Q}$ such that $Q_1 \subset Q$
$H^{\ell}(Q_1) = 0$ whenever $0 < \ell < p$.

If Y admits an n-mean, then given any $Q \in \mathcal{Q}$, there exists $Q_1 \in \mathcal{Q}$ such that $H^p(Q_1)$ is n-divisible.

Proof:

Assume the hypotheses, let $m : Y^n \to Y$ be an n-mean and suppose $Q \in \mathcal{Q}$. Now there exists an open set U in Y containing x_o such that $m(U^n) \subset P$ and by (1) and (2), there exists $Q_1 \in \mathcal{Q}$ such that $Q_1 \subset U \cap Q$ and $H^{\ell}(Q_1) = 0$ for $0 < \ell < p$; hence, $m(Q_1^n) \subset P$. Letting $i: Q_1 \subset P$ and $k : Q_1 \subset X$, we see that $i^* = k^* \cdot j^* : H^p(P) \to H^p(Q_1)$ is an epimorphism. But according to (2) and Proposition 4.8, $i^* H^p(P)$ is n-divisible in $H^p(Q_1)$, so we conclude that $H^p(Q_1)$ is n-divisible.

Corollary 4.10

Let $p \geq 2$ and for each integer $t \geq 1$, let

$C_t = \{x \in R^p \mid \| x - (\frac{1}{t},0,0,\ldots,0) \| = \frac{1}{t}\}$. If $X = \bigcup_{t \geq 1} C_t$, then for any $n \geq 2$, the cone $C(X)$ over X does not admit an n-mean.

Proof:

Let $Y = C(X)$, let v denote the vertex of Y, identify X with $\{(\overline{x,0}) \mid x \in X\} \subset Y$ and let $x_o = (0,0,0,\ldots,0) \in X \subset Y$. Let P be the open set $Y \setminus \{v\}$ (which contains x_o) and for each integer $t_o \geq 1$, let $Q_{t_o} = \bigcup_{t=t_o}^{\infty} C_t$. Then, since $\mathcal{Q} = \{Q_{t_o} \mid t_o \geq 1\}$ is a tower of continua satisfying the hypotheses of Proposition 4.9, and since $H^p(Q_{t_o})$ is n-divisible for no $n \geq 2$, $Y = C(X)$ cannot admit an n-mean for $n \geq 2$.

The next proposition and its corollary, due to Sigmon, are proven elegantly and directly in [19]. The proof of Proposition 4.11 offered here will be based on Proposition 4.7.

Proposition 4.11

Let X be a continuum and R a principal ideal domain such that $H^p(X;R)$ is torsion free as an R-module, for each $p \geq 1$. If X admits an n-mean, then $H^p(X;R)$ is uniquely n-divisible for each $p \geq 1$.

Proof:

We shall induct on p. If $p = 1$ and $h \in H^1(X;R) = H^1(X)$, then by Proposition 4.7 (with $X = Q = P$ and $m : X^n \to X$) we have

$$h = \Delta^* m^*(h) = m_1^*(h) + m_2^*(h) + \ldots + m_n^*(h) = n \cdot m_1^*(h) \in n \cdot H^1(X).$$

Therefore (since $H^1(X)$ is torsion free), there is a unique $h' \in H^1(X)$ such that $h = nh'$. Now suppose $p > 1$ and $H^j(X)$ is uniquely n-divisible for $j < p$. If $h \in H^p(X)$, then letting Σ^p denote

$$\sum_{\substack{i_1+\ldots+i_n=p \\ 0 \leq i_1, i_2, \ldots i_n < p}} H^{i_1}(X) \otimes H^{i_2}(X) \otimes \ldots \otimes H^{i_n}(X),$$ we have (4.7 again)

$h = \Delta^* m^*(h) = m_1^*(h) + \ldots + m_n^*(h) + \Delta^*\mu(\theta)$ where $\theta \in \Sigma^p$. Now, since each $H^j(X)$ is uniquely n-divisible $(j=1,2,\ldots,p-1)$, so is Σ^p; hence, there exists $\theta' \in \Sigma^p$ such that $\theta = n\theta'$. But then

$h = n \cdot m_1^*(h) + \Delta^*\mu(\theta) = n[m_1^*(h) + \Delta^*\mu(\theta')] \in nH^p(X)$, and we conclude that $H^p(X)$ is uniquely n-divisible.

Corollary 4.12

If a compact T_2 space X admits an n-mean, then $H^p(X;Z_n) = 0$ for each $p \geq 1$.

Proof:

We begin with three observations: first, it is sufficient to show that $H^p(C;Z_n) = 0$ for each component C of X and hence (since each component of a space admitting an n-mean must also admit an n-mean), it is sufficient to show the corollary is true if X is assumed connected. Second, if a space Y admits an n-mean and if p is any factor of n, then Y admits a p-mean. (See Problem 1 in the last section). Third, if $0 \to G' \to G \to G'' \to 0$ is an exact sequence of groups, then there is an exact sequence

$$\ldots \to H^{p-1}(X;G'') \to H^p(X;G') \to H^p(X;G) \to H^p(X;G'') \to H^{p+1}(X;G') \to \ldots .$$

We suppose X is a continuum which admits an n-mean and write $n = p_1 \ldots p_r$ where each p_i is prime. The proof will be by induction on r. If $r=1$, then $n = p_1$ is prime so that Z_n is a field; consequently, $H^p(X;Z_n)$ is a vector space over Z_n and hence is torsion free as a Z_n-module. Since each element of $H^p(X;Z_n)$ has additive order n, and since (by 4.11), it is n-divisible, we conclude that $H^p(X;Z_n) = 0$.

Now for the induction step, suppose $r > 1$ is such that: If $q_1 q_2 \ldots q_t$ is a product of primes where $t < r$ and X is a continuum admitting a $q_1 q_2 \ldots q_t$-mean, then $H^p(X;Z_{q_1 \ldots q_t}) = 0$. Then if a continuum X admits an $n = p_1 p_2 \ldots p_r$-mean, then the exact sequence

$$0 \to Z_{p_1 \ldots p_{r-1}} \to Z_n \to Z_{p_r} \to 0$$

induces an exact sequence

$$\ldots \to H^p(X;Z_{p_1 \ldots p_{r-1}}) \to H^p(X;Z_n) \to H^p(X;Z_{p_r}) \to H^{p+1}(X;Z_{p_1 \ldots p_{r-1}}) \to \ldots .$$

But by the induction supposition, $H^p(X;Z_{p_1 \ldots p_{r-1}}) = 0 = H^p(X;Z_{p_r})$ so that $H^p(X;Z_n) = 0$. This completes the proof.

<u>Proposition 4.13</u>

If the compact T_2 space X admits an n-mean, then $H^p(X;Z)$ is uniquely n-divisible for each $p \geq 1$.

<u>Proof</u>:

If $p \geq 1$, then $H^p(X;Z_n) = 0$, so we conclude from the exact sequence below (universal coefficient theorem)

$$0 \to H^p(X;Z) \odot Z_n \to H^p(X;Z_n) \to H^{p+1}(X;Z) * Z_n \to 0$$

that
$$0 = H^p(X;Z) \odot Z_n = H^p(X;Z) / nH^p(X;Z) \quad \text{and that}$$
$$0 = H^{p+1}(X;Z) * Z_n = \{h \in H^{p+1}(Z;Z) \mid nh = 0\}.$$ Hence, (using Proposition 4.5), we conclude that $H^p(X;Z)$ is uniquely n-divisible.

This result has the curious consequence that the contravariant functor $H^p(-;Z)$ (for any $p \geq 1$) carries an <u>injective</u> object in the

category of compact T_2 spaces to an <u>in</u>jective object in the category of Abelian groups.

As another corollary to Proposition 4.7, we have the following.

<u>Proposition 4.14</u>

If the continuum X admits an idempotent continuous multiplication $m : X \times X \to X$ having a left and right zero, then X is acyclic.

<u>Proof:</u>

To prove that $H^p(X;Z) = 0$ for $p \geq 1$, we proceed by induction on p. $\Delta : X \to X \times X$ will denote the diagonal map. If $p = 1$ and $h \in H^1(X)$, then by Proposition 4.7, $h = \Delta^* m^* h = \Delta^*(m_1^*(h) \times 1 + 1 \times m_2^*(h))$ $= m_1^*(h) \cup 1 + 1 \cup m_2^*(h) = m_1^*(h) + m_2^*(h)$ where $m_1, m_2 : X \to X$ may be taken to be $m_1(x) = m(x,z)$ and $m_2(x) = m(z,x)$ where $z \in X$ is arbitrary. But if z is the zero, then $m_1^* = 0 = m_2^*$ so that $h = 0$. Thus $H^1(X) = 0$.

Now suppose $p > 1$ and $H^i(X) = 0$ whenever $0 < i < p$. Then again we have $h = m_1^*(h) + m_2^*(h) = 0$ for any $h \in H^p(X)$; hence $H^p(X) = 0$.

Proposition 4.14 above is reminiscent of a result of Wallace [21]: If a continuum S admits a semigroup with zero and if the set of idempotents E is commutative and satisfies $S = ES = SE$, then S is acyclic.

SECTION 5 PROBLEMS

1. <u>If X is a space, then X admits an mn-mean if and only if X admits both an m-mean and an n-mean(?).</u> We have seen that this is true if X is a solenoid (Corollary 3.6). Furthermore, if X admits an mn-mean $f : X^{mn} \to X$, then $f \circ g$ is an n-mean where $g : X^n \to X^{mn}$ is defined by $g(x_1, \ldots x_n) = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_n, \ldots, x_n)$ each factor being repeated m times.

2. <u>If X is a locally connected continuum which admits an n-mean for each $n \geq 1$, then X is acyclic (?).</u> It has been seen that the local connectedness hypothesis cannot be removed (the rational solenoid Σ_ω) and that a polyhedron which admits any n-mean must be contractible [10].

3. <u>If X is a locally connected continuum in R^p $(p \geq 2)$ which admits a 2-mean, then must X admit all n-means? must X be an absolute retract?</u>

If $p = 2$, both conclusions are true (Proposition 2.8). Also, there exist (contractible) locally connected continua in R^p $(p > 2)$ which

(a) admit all n-means (I^p),

or (b) admit no n-means (Corollary 4.10).

There exist non-locally connected (in fact, indecomposable) continua in R^3 which

(c) admit no n-means ($\Sigma_{(2,3,5,7,11,\ldots)}$),

(d) admit some but not all n-means (Σ_n),

or (e) admit all n-means (Σ_ω).

4. If M is an n-mean on a solenoid Σ_a, then must M be homomorphic? We have seen (Proposition 3.5) that the solenoid Σ_a admits an n-mean if and only if it admits the homomorphic n-mean.

5. If X is a circle-like metric continuum which admits some n-mean, then X is a solenoid(?). For definitions, see Mahavier [15] in which he proves: if X is a circle-like metric continuum which supports a semigroup with identity, then X is a solenoid.

6. If X is a continuum which fails to admit an n-mean, then under what conditions does the cone $C(X)$ admit an n-mean? S^1 admits no n-mean, while $C(S^1)$ admits all n-means. The continuum X of Corollary 4.10 admits no n-mean, while $C(X)$ admits none. If a space X admits an n-mean, then so does $C(X)$. If X is a space, then $C(X)$ admits an n-mean if and only if $C(C(X))$ admits an n-mean. (Because a product space admits an n-mean if and only if each factor admits an n-mean, while $C(C(Y)) = C(Y) \times I$ for any space Y [18].)

7. Under what conditions on a continuum X and integer $k \geq 2$ does there exist a continuous function $f_k : \mathcal{F}_{2k}(X) \to \mathcal{F}_k(X \times X)$ such that $f_k(\{x\}) = \{(x,x)\}$ for each $x \in X$?

If such a function f_k exists and if X also admits a uniform k-mean, $r : \mathcal{F}_k(X) \to X$, then letting $m : X^2 \to X$ be the induced 2-mean (Proposition 2.3), there is a continuous function $M : 2^{X \times X} \to 2^X$ defined by $M(C) = \{m(c,c') \mid (c,c') \in C\}$ whose restriction $M' : \mathcal{F}_k(X \times X) \to \mathcal{F}_k(X)$ can be used to construct the uniform 2k-mean

$$r \circ M' \circ f_k : \mathcal{F}_{2k}(X) \to \mathcal{F}_k(X \times X) \to \mathcal{F}_k(X) \to X.$$

In particular, if X admits a function f_2 and X admits a 2-mean, then X admits a 3-mean and a 4-mean. If X admits a function f_k for each $k \in \{2,4,8,16,\ldots\}$ and X admits a 2-mean, then X admits a uniform n-mean for each $n \geq 2$.

If X is an absolute retract, then such an f_k exists for each $k \geq 2$.

8. Is there a space X which admits all n-means but which is not a retract of $\mathcal{F}(X) = \bigcup_{n=1}^{\infty} \mathcal{F}_n(X)$?

9. Under what conditions on a compact T_2 space does a retraction $r : \mathcal{F}(X) \to X$ induce an order on X in such a way that an infimum map $2^X \to X$ extends r?

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BIOGRAPHICAL SKETCH

Arthur Crummer was born (1943) and raised in the United States by fundamentalist Protestant parents and takes pleasure from handball, fishing, travel, music, gardening, bicycling, people and mathematics. He shares and celebrates his life with his wife Mary, son Thor, dog Nemo, and a few friends.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertaion for the degree of Doctor of Philosophy.

Philip Bacon

Philip Bacon, Chairman
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

A. D. Wallace
Graduate Research Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

James Keesling

James Keesling
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Kermit Sigmon

Kermit Sigmon
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Richard Irey
Professor of Mechanical ~~Engineering~~

This dissertation was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

June, 1971

Dean, College of Arts and Sciences

Dean, Graduate School

#597 Lee Man
126)